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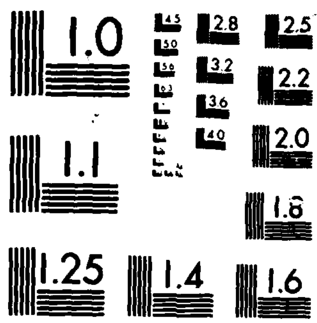

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UNCERTAINTIES IN ENGINEERING DESIGN: MATHEMATICAL THEORY AND NUMERICAL EXPERIENCE

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MATHEMATICAL THEORY AND NUMERICAL EXPERIENCE

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ABSTRACT

The paper addresses the question of the reliability of engineering computations. It brings a set of paradoxical, unexpected results which shows that the common practice can lead to unreliable results and conclusions. The theory and implementation of the analysis of elasticity problems with stochastic input data (loads, domain, coefficients) are outlined. Numerical examples illustrate the ideas and results.

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1. INTRODUCTION

Shape optimization in the structure mechanics became to be in the center of the research and applications. Many papers and books dealing with this subject appeared and special conferences dealt with these problems. The research is directed toward theoretical questions as existence and characterization of the optimal design, bounds for the optimized values, numerical treatment of the optimal design problems, etc. We will not mention here the recent vast available literature. We mention only [10] [13] [18] [21] as examples.

addresses

We will address in this paper the problem of the reliability of the conclusions based on the computational analysis and their relation to the problems of the optimal design.

By reliability we mean here that the conclusions sufficiently accurately describe the physical reality.

The problem of optimal design consists--in principle--in the comparison of the solutions of states from the set S of admissible states (for example, solution of the problems from the set of admissible domains) and the selection of the "optimal" state (e.g. the domain) for the engineering design. It is obvious that such a selection can be successful only if the solution of every state is uniformly reliable with respect to the entire set S. This requirement creates a serious difficulty because we are used to solve in practice numerically the simplified mathematical formulation of the problem and have experience only with small limited set of practical problems. Hence, it is essential to analyze and to be explicitly aware of the assumptions used in the derivation of the model and its numerical treatment and the limitations we have to deal with.

From what we said, it is obvious, that we have to focus on the reliability of the analysis of the single states and its uniformity over the (entire) set S of admissible states. This has always to be the starting point of the assesment of the validity of the optimal design.

The reliability of the computational treatment of the single states depends on

a) mathematical model

b) reliability of the input data

c) reliability of the numerical treatment.

These three aspects are of course closely related.

In this paper we will address the questions not in general, but on few concrete engineering examples. We restrict ourselves to the examples which are relatively simple from the engineering (although not mathematical) point of view, to present the ideas in a most clear way.

2. THE PROBLEM OF A TUBE WITH A STIFFENED SURFACE

Let us consider the problem of a tube (plane strain) with stiffened outer surface. We shall assume that the stiffener is bending free. We are in general interested in the design resp. the influence of the changes of the outer surface Γ_0. The scheme of the problem is shown in Fig. 2.1.

We will formulate the problem as <u>linear elasticity</u> (plane strain) problem. The formulation is the standard one, based on the minimization of the (cummulative) energy of the tube and the (tension) energy of the reinforcement on Γ_0.

2.1. The reliability of the mathematical model

We will analyze the case when the outer boundary Γ_0 is a regular m-polygon and Γ_1 is a concentric circle. See Fig. 2.2. The domain is denoted

by Ω_m. The circular domain Ω_0 (see Fig. 2.3) is obviously the limiting case when $m \to \infty$. Assuming that the unit hydrostatic pressure is given on the inner boundary and that the stiffener is infinitely rigid in tension, the problem of linear elasticity can be formulated on one sector only with the boundary conditions shown in Fig. 2.4. On the sides A-B and C-D the symmetry conditions are prescribed. On the side A-D the boundary conditions describe the behaviour of the stiffened side and on the side B-C the tractions are prescribed.

Denote by (u_m, v_m) resp. (u_0, v_0) the solution (displacement) on Ω_m resp. Ω_0.

From the physical grounds we have to expect that $(u_m, u_m) \to (u_0, v_0)$ as $m \to \infty$. If $(u_m, v_m) \not\to (u_0, v_0)$, we have to have doubts about the reliability of the model.

We have

THEOREM 2.1.

$$\lim_{m \to \infty} (u_m, u_m) = (u_\infty, v_\infty) \neq (u_0, v_0). \tag{2.1}$$ ■

For the analysis of the dependence of the solution on small changes of the domains we refer e.g. to [1] [3] [4].

Theorem 2.1 shows that the used <u>model of linear elasticity is unreliable</u> at least if the outer boundary is not smooth and hence it <u>practically cannot be used</u> for optimal design when the admissible domains have not sufficiently smooth boundary. We see here that the uniform reliability (with respect to m) is clearly violated.

The limiting solutions (u_∞, v_∞) and (u_0, v_0) can be found.

THEOREM 2.2. The solutions (u_∞, v_∞) and (u_0, v_0) are radially symmetric. Denoting by σ_r resp. σ_θ the stresses in polar coordinates we get:

$$\sigma_r = \frac{A}{r^2} + B, \qquad \sigma_\theta = -\frac{A}{r^2} + B \tag{2.2}$$

with

$$A_\infty = a^2, \qquad B_\infty = 0 \tag{2.3}$$

$$A_0 = \frac{(1-\nu)a^2b^2}{(1-\nu)b^2 + (1+\nu)a^2}, \qquad B_0 = -\frac{(1+\nu)a^2}{(1-\nu)b^2 + (1+\nu)a^2}. \quad \blacksquare \tag{2.4}$$

Table 2.1 gives the values of the stresses σ_r and σ_θ on the line C-D for the solution (u_∞, v_∞) and (u_0, v_0) when $a = 0.3$, $b = 1.0$ and $\nu = 0.3$. We see clearly that the solutions (u_∞, v_∞) and (u_0, v_0) are essentially different.

TABLE 2.1

r	(u_∞, v_∞)		(u_0, v_0)	
	σ_r	σ_θ	σ_r	σ_θ
0.3	-1.000	1.000	-1.000	0.7135
0.4	-0.5635	0.5625	-0.6251	0.3387
0.5	-0.3600	0.3600	-0.4516	0.1652
0.6	-0.2500	0.2500	-0.3574	0.0710
0.7	-0.1836	0.1836	-0.3005	-0.0142
0.8	-0.1406	0.1406	-0.2637	-0.0227
0.9	-0.1111	0.1111	-0.2384	-0.0480
1.0	-0.0900	0.0900	-0.2208	-0.0609

Before discussing the probable reason for this paradox, let us mention

THEOREM 2.3. Let (u_m, v_m) be the solution on Ω_m (i.e., m-sided polygon), $m > 4$. Then the solution has a singularity in the neighborhood of the point A (vertex of the polygon) (see Fig. 2.2 and 2.4) and

$$(2.5) \qquad \begin{matrix} u_m \\ v_m \end{matrix} = r^{\frac{2}{m-2}} \begin{pmatrix} \varphi_m(\theta) \\ \psi_m(\theta) \end{pmatrix} + \text{higher order terms}$$

where (r,θ) are the polar coordinates with the origin in A and φ_m, ψ_m are smooth functions in θ. ∎

Theorem 2.3 shows that the solution has a strong singularity in the neighborhood of A and the strains and stresses are there unbounded. This obviously violates the basic assumptions of the linear elasticity model and has unexpected consequences.

We are making the following conjecture (unproven):

If $(\tilde{u}_m, \tilde{v}_m)$ is the solution of a nonlinear problem, then $\lim(\tilde{u}_m, \tilde{v}_m) \to (\tilde{u}_0, \tilde{v}_0)$.

This leads to the following conclusion. If the set of admissible domains has unsmooth outer boundary, then it is necessary to use nonlinear theory of elasticity in the optimal design problems. The linear elasticity leads to unreliable results and conclusions.

2.2. The reliability of the numerical solution

As we have seen in Theorem 2.3, the solution has a very strong singularity (note that in the case of a crack the singularity of the solution is $r^{1/2}$) which makes computation very difficult for larger m. The computation we present has been make by the code PROBE which uses p and h-p version of the finite element method. See [23] [24]. For the theoretical aspects we refer to [6] [7] [14]. The mesh has to be strongly refined in the area of the

singularity if reliable results have to be obtained. See [14] [22]. For the p-version (i.e. when there is no strong refinement at A) the energy norm of the error is $\|e\| \approx Cp^{-4/(m-2)}$. See [6] [9]. For the h-version without properly refined mesh, the situation is still worse. For the properly refined mesh the rate of convergence in the first phase (p not large) is exponential. See [14]. The mesh we used is shown in Fig. 2.5 ($a = 0.3$, $b = 1$, $\nu = 0.3$).

In Table 2.2 we show the stresses in the points P, Q, R, S (see Fig. 2.5) for $m = 8, 16, 32$ for various degrees p of elements. We see that the solution is close to the limiting value of $m = \infty$. Although $\bar{\sigma} = \sigma_{x\infty} + \sigma_{y\infty} = 0$ we see that $\bar{\sigma}$ deteriorates from $m = 16$ to $m = 32$. The reason is that the quality of the numerical solution deteriorates with $m \to \infty$. This deterioration is, for example, visible from the Table 2.3 where the computed strain energy for various p and m is given. We see clearly a much larger change in the energy for $m = 32$ than for $m = 8$ when increasing the degree p. This indicates much larger error for $m = 32$ than for $m = 8$. The strength of the singularity is $r^{2/(m-2)}$ which is so strong that without special care no reasonable accuracy can be achieved for $m = 32$. For $m = 16$ the error in the energy norm is expected to be 2-4%, and 7-10% for $m = 32$ in our computations.

Table 2.4 shows the values of the maximal principle stress in the points B-F and $\bar{B}$-$\bar{F}$ (see Fig. 2.5). We clearly see that the stresses are very large in the neighborhood of the vertices and with $m \to \infty$ the stresses are increasing (because the strength of the singularity is increasing). This also clearly indicates the likely reason for the paradox we mentioned above.

We see that not only the mathematical model but also the quality numerical solution is very nonuniform with respect to small changes of boundary (which does not have sufficient smoothness).

TABLE 2.2

POINT	r	p	m = 8			m = 16			m = 32			m = ∞
			σ_x	σ_y	τ_{xy}	σ_x	σ_y	τ_{xy}	σ_x	σ_y	τ_{xy}	σ_y
P	0.3	8	+0.9992	-1.000	+1.501 -5	+0.9953	-1.000	-0.579 -4	+0.9634	-1.0000	-0.297 -3	-1.000
		7	+0.9995	-0.9999	+0.110 -3	+0.9960	-0.9998	-0.213 -3	+0.9629	-0.9999	+0.520 -3	EXACT
		6	+0.9990	-0.9999	-0.491 -3	+0.9940	-1.000	-0.552 -3	+0.9600	-0.9999	-0.893 -3	
		5	+1.001	-0.9998	+0.140 -2	+0.9964	-0.9987	+0.134 -2	+0.9600	-0.9987	+0.163 -2	
Q	0.5	8	+0.3564	-0.3567	-0.576 -4	-0.3571	-0.3641	-0.634 -5	+0.3353	-0.3715	+0.108 -3	-0.3600
		7	+0.3569	-0.3566	-0.108 -4	+0.3571	-0.3612	+0.195 -4	+0.3344	-0.3719	+0.108 -4	EXACT
		6	+0.3566	-0.3562	+0.403 -3	+0.3556	-0.3617	+0.185 -3	+0.3317	-0.3734	+0.594 -4	
		5	+0.3548	-0.3568	-0.160 -2	+0.3554	-0.3631	-0.902 -3	+0.3328	-0.3735	+0.396 -4	
R	0.7	8	+0.1578	-0.1558	+0.527 -3	+0.1775	-0.1759	+0.119 -3	+0.1634	-0.1814	-0.483 -2	-0.1836
		7	+0.1598	-0.1643	-0.137 -3	+0.1814	-0.1958	+0.700 -3	+0.1581	-0.2216	+0.679 -2	EXACT
		6	+0.1588	-0.1484	-0.981 -3	+0.1765	-0.1670	-0.218 -2	+0.1672	-0.1705	-0.904 -2	
		5	+0.1550	-0.1756	+0.262 -2	+0.1761	-0.2083	+0.456 -2	+0.1472	-0.2353	+0.117 -1	
S	0.8	8	+0.0891	-0.0888	+0.107 -2	+0.1318	-0.1338	+0.868 -3	+0.1232	-0.1556	-0.128 -3	-0.1406
		7	+0.0884	-0.0879	-0.169 -2	+0.1278	-0.1302	-0.120 -2	+0.1162	-0.1518	-0.449 -3	EXACT
		6	+0.0846	-0.0853	+0.187 -2	+0.1214	-0.1345	+0.234 -2	+0.1017	-0.1655	+0.981 -3	
		5	+0.0934	-0.0893	-0.207 -2	+0.1446	-0.1411	-0.236 -2	+0.1447	-0.1608	-0.204 -2	

TABLE 2.3

p	m = 8	m = 16	m = 32
8	0.229725	0.114518-1	0.559748-2
7	0.229723	0.114491-1	0.559199-2
6	0.229719	0.114456-1	0.558305-2
5	0.229705	0.114402-1	0.557552-2
4	0.229651	0.114298-1	0.556082-2

TABLE 2.4

POINT	p	Maximal principal stress		
		m = 8	m = 16	m = 32
F	8	-0.3807+0	-0.3758+0	-0.3624+0
	7	-0.3814+0	-0.3767+0	-0.3670+0
	6	-0.3802+0	-0.3716+0	-0.3494+0
E	8	-0.1360+1	-0.1907+1	-0.2043+1
	7	-0.1358+1	-0.1916+1	-0.2076+1
	6	-0.1355+1	-0.1879+1	-0.1990+1
D	8	-0.4805+1	-0.9498+1	-0.1150+2
	7	-0.4810+1	-0.9502+1	-0.1153+2
	6	-0.4748+1	-0.9394+1	-0.1119+2
C	8	-0.1709+2	-0.4695+2	-0.6393+2
	7	-0.1709+2	-0.4683+2	-0.6383+2
	6	-0.1769+2	-0.4585+2	-0.6176+2
B	8	-0.5296+2	-0.1764+3	-0.2722+3
	7	-0.6709+2	-0.2762+3	-0.4340+3
	6	-0.4972+2	-0.1442+3	-0.1931+3
F	8	-0.3851+0	-0.4044+0	-0.5062+2
	7	-0.3855+0	-0.4048+0	-0.4998+0
	6	-0.3821+0	-0.3947+0	-0.4767+0
E	8	-0.1356+1	-0.2111+1	-0.3358+1
	7	-0.1358+1	-0.2135+1	-0.3427+1
	6	-0.1335+1	-0.2083+1	-0.3351+1
D	8	-0.4818+1	-0.1130+2	-0.2136+2
	7	-0.4825+1	-0.1147+2	-0.2184+2
	6	-0.4748+1	-0.1121+2	-0.2136+2
C	8	-0.1740+2	-0.6362+2	-0.1400+3
	7	-0.1747+2	-0.6459+2	-0.1435+3
	6	-0.1722+2	-0.6349+2	-0.1408+3
B	8	-0.6361+2	-0.3590+3	-0.9021+3
	7	-0.6481+2	-0.3824+3	-0.9606+3
	6	-0.6437+2	-0.3636+3	-0.8924+3

3. THE PROBLEM OF THE PLATES AND SHELLS

In Section 2 we addressed the problem of the reliability of the linear elasticity model. Models of plates and shells are two dimensional although obviously the original problem is three dimensional. Hence, we will assume that the three-dimensional linear elasticity formulation is reliable and will analyze only the effects of the dimensional reduction from 3 to 2 dimensions and the implication for the optimal design.

3.1. The problem of the simply supported plate

Let us, for simplicity, assume that we are concerned with the case when the Poisson ratio $\nu = 0$. The plate problem (with uniform thickness h) can be formulated in various ways. Let us mention the "projection method" when we assume an "ansatz" and use it in the variational principle by minimizing the energy. This approach is sometimes called Kantorowich method (see [15]). Denoting u, v, w the displacement components, we shall consider two "ansatzes":

1) The K (Kirchhoff) model

$$u(x,y,z) = -z \frac{\partial w}{\partial x}(x,y), \tag{3.1a}$$

$$v(x,y,z) = -z \frac{\partial w}{\partial y}(x,y), \tag{3.1b}$$

$$w(x,y,z) = w(x,y). \tag{3.1c}$$

Using this ansatz in the potential energy principle we get the usual formulation

$$EI \, \Delta\Delta \, w = f \tag{3.2}$$

and the simple support is obtained by minimization of the energy with the <u>only</u> constraint $w = 0$ on the boundary Γ of the domain.

2) The R-M (Reissner-Mindlin) model

$$u(x,y,z) = -z\varphi(x,y) \tag{3.3a}$$

$$v(x,y,z) = -z\psi(x,y) \tag{3.3b}$$

$$w(x,y,z) = w(x,y) \tag{3.3c}$$

Utilizing (3.3) in the expression for the three-dimensional potential energy and imposing the (<u>only</u>) constraint $w = 0$ at Γ we obtain a system of three differential equations of second order in contrast to one equation of fourth order in the K-model.

The dimensional reduction has been analyzed in the asymptotic way when $h \to 0$ and the solution is smooth. See e.g. [11] [12] [20]. In this asymptotic frame we <u>cannot</u> distinguish between the two mentioned models. Physically the R-M model is taking into account the shear stresses while the K model neglects them. Let us once more assume that Ω_m is the regular m-polygon inscribed in the circle of radius a, Ω_0 be the circle with radius a and let us consider the problem of uniformly loaded (by load p) simply supported plate.

Denote by w_m resp. w_0 and $(\varphi_m, \psi_m, \bar{w}_m)$ resp. $(\varphi_0, \psi_0, \bar{w}_0)$ the solution of the K and K-M model on Ω_m and Ω_0. Then we have

THEOREM 3.1.

$$w_m \to w_\infty \neq w_0 \tag{3.4a}$$

$$(\varphi_m, \psi_m, \bar{w}_m) \to (\varphi_\infty, \psi_\infty, \bar{w}_\infty) = (\varphi_0, \psi_0, \bar{w}_0). \tag{3.4b}$$

See [3] [5]. ∎

We can compute the limiting solution w_∞ and $(\varphi_\infty, \psi_\infty, \bar{w}_\infty)$ analytically. For the K-model we have

$$(3.5a) \qquad w_\infty(0,0) = -\frac{3}{64} a^4 \frac{p}{EI}$$

$$(3.5b) \qquad w_0(0,0) = -\frac{5}{64} a^4 \frac{p}{EI}$$

and for the R-M model we have

$$(3.5c) \qquad w_\infty(0,0) = w_0(0,0) = -\frac{5}{64} a^4 \frac{p}{EI} - \frac{pa^2}{EF}$$

where $I = h^3/12$, $F = h$ are the moment of inertia and the thickness, respectively, and E is the modul of elasticity.

Theorem 3.1 and the formulae 3.5a-3.5c show that the effects of the shear stress in the neighborhood of the corners are essential. Although we discussed only the problem of the polygon plate, the analysis (see [3] [5]) covers much more general solutions and clearly point to the following conclussion:

The optimal design of a plate has to be based on the R-M and not the K-model.

We will not discuss here the reliability of the numerical treatment. Similar but more complicated situation occurs in the case of the shells.

3.2. The problem of the plate with a variable thickness

Let us consider a plate with variable thickness. If the thickness is very slowly varying with respect to the thickness of the plate, then the derivation (dimensional reduction) can be made in the same way as for the

constant thickness. Nevertheless, if the thickness is varying rapidly, then the classical derivation is not valid. Recently a theory has been developed (see [16] [17]) which shows important relation between the thickness and thickness variation and which strongly influences the reliability of the mathematical model. We will show it in the most simple setting. Consider the stiffened plate shown in Fig. 3.1. The main idea of the classical plate derivation is to consider the limiting process $\varepsilon \to 0$ and apply the results for $\varepsilon > 0$.

We can assume that $a = C_1\varepsilon^{\lambda_1}$, $b = C_2\varepsilon^{\lambda_2}$ and consider the limiting process $\varepsilon \to 0$. In [17], $\lambda_1 = \lambda_2 = \lambda$ is assumed and it is shown that we get different model for $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$.

In the case of $\lambda < 1$ the stiffners are far apart when $\varepsilon \to 0$, in the case $\lambda > 1$ they are close together. In all three cases the dimensional reduction leads to the plate formulation with effective coefficients depending on the value of λ. This example shows that optimal design based on one model, say, $\lambda < 1$ for fixed but small thickness can lead to a design when the model is not valid (reliable) anymore. Using a proper model for this design and redoing the optimal design once more, we can once more get out of the range of the reliability of the model. Hence, we have to consider here <u>simultaneous design optimization and the model selection</u>. For important aspects of this problem directly related to the optimal design, we refer to [17].

4. THE PROBLEM OF A SUPPORTED CONSTRUCTION

Let us consider the optimal design of a supported construction (see Fig. 4.1). The problem is how to model the support in the point B. To show the difficulty, let us consider the problem shown in Fig. 4.2 and solve the linear

elasticity (plane stress, $\nu = 0.3$) problem. The standard finite element modeling is to make constraint $v = 0$ at the node located in the support. This modeling is incorrect because the solution strongly depends on the finite element method.

Let M be the moment at the side A-A′ and M_N is the moment computed by the finite element method. Assume that the size h of the maximal element $h_N = \max h \to 0$ as $N \to \infty$. We have

THEOREM 4.1.

$$\lim_{N\to\infty} M_N = M_0 \tag{4.1}$$

where M_0 is the moment when there is no support (and hence M_0 can be analytically computed).

Theorem 4.1 shows that by selecting different meshes we can get completely different results and hence optimal design will strongly depend on the used mesh. In fact, the situation is still more complicated because $M_N \to M_0$ slowly and we have no means to establish how reliable the solution is.

Before discussing this effect, let us show the computation by the code PROBE. The used mesh is shown in Fig. 4.3. There is refinement in the neighborhood of A-A′ and especially strong refinements is in the neighborhood of B (see Fig. 4.4). We did use two meshes A_4, with smallest ring of the radius a_4, and A_5 with the radius a_5. Table 4.1 shows the moments on A-A′ and Table 4.2 shows the displacement v in C. Although the moments and the displacement are significantly smaller than that of the unsupported beam, the mesh dependence is obvious. Note that the difference between the values obtained by the mesh A_4 and A_5 is nearly independent of p. The reason for the effects we have shown is that the support is not correctly modeled. The

reaction is a point force which leads to the infinite energy and a infinite displacement in the point of the reaction. The infinite displacement at the reaction point can be seen from the analytical solution on half plane with concentrated load. Hence, reaction has to be zero and we obtain the solution of an unsupported beam.

TABLE 4.1

p	Mesh A_4	Mesh A_5
3	1.875	1.918
4	1.909	1.954
5	1.921	1.965
6	1.931	1.976
7	1.939	1.984
8	1.946	1.991

TABLE 4.2

p	Mesh A_4	Mesh A_5
1	- 7.92	-10.68
2	-10.94	-14.91
3	-13.14	-17.31
4	-14.56	-18.77
5	-15.64	-19.85
6	-16.37	-20.58
7	-17.38	-21.21
8	-17.56	-21.76

In Table 4.3 we show the displacements in the points B_i and $\bar{B}_i$ computed by the mesh A_4 and A_5. Realizing that the distance between B and B_5 is $0.75\ 10^{-4}$ and the constraint in B is $v = 0$, we see numerically the effect mentioned above. This clearly shows that the mathematical model of the supported beam is unrealiable because it does not distinguish between supported and unsupported beam. Hence, a more sophisticated model of

TABLE 4.3

POINT	p	Mesh A_4	Mesh A_5	POINT	Mesh A_4	Mesh A_5
B_5	8 7 6 5		- 9.189 - 8.777 - 8.313 - 7.764	$\bar{B}_5$		- 9.188 - 8.777 - 8.312 - 7.763
B_4	8 7 6 5	- 9.231 - 8.818 - 8.352 - 7.801	-12.36 -11.95 -11.49 -10.94	$\bar{B}_4$	- 9.226 - 9.831 - 7.795 - 7.123	-12.36 -11.95 -11.48 -10.94
B_3	8 7 6 5	-12.43 -12.02 -11.55 -11.00	-15.54 -15.14 -14.67 -14.13	$\bar{B}_3$	-12.39 -11.98 -11.57 -10.97	-15.51 -15.11 -14.64 -14.10
B_2	8 7 6 5	-15.72 -15.31 -14.85 -14.31	-18.80 -18.40 -17.94 -17.40	$\bar{B}_2$	-15.48 -15.07 -14.05 -13.37	-18.61 -18.20 -17.73 -17.19
B_1	8 7 6 5	-19.60 -19.22 -18.78 -18.27	-22.52 -22.14 -21.17 -21.20	$\bar{B}_1$	-18.01 -17.58 -16.52 -15.80	-21.27 -20.84 -20.36 -19.79

the support is needed. Nevertheless, we will not discuss here the question of a reliable model. [Usually it is claimed that a concrete not strongly refined mesh models the support. It is obvious that without a reference to the proper formulation of the support the claim has no firm meaning.]

5. THE PROBLEM OF THE STOCHASTIC INPUT DATA

The basic input data describing the elasticity problems are: the domain, the material propertis and the loads. Assume now that the data are stochastic functions. For example, the boundary of the domain can be

described by a stochastic function which expresses the uncertainty of fabrication. Then the solution is also a stochastic function. In addition, the failure criterium which can be basis for the optimal design is always a stochastic one. Hence, we have combine both stochastic characters to get desired information. Because the uncertainty of the input data, the dispersion of the results can be significant. Recently we developed a theory of the solution with stochastic input data (see [2] [19] and forthcoming papers) and their numerical tratment by the finite element method. The implementation is based on the code PROBE mentioned earlier.

5.1. The case of the stochastic load

Let us consider the container of the form shown in Fig. 5.1. The side AB is loaded by a horizontal stochastic function $X_x = \Lambda(y,\omega)$, $0 < y < H$ and we will assume that $\overline{X_x(y,\omega)} = 1$ where we denoted by $\overline{X_x(y,\omega)}$ the mean. The correlation function is assumed to be

$$K(y_1,y_2) = 0.1^2 e^{-\alpha|y_1-y_2|}. \qquad (5.1)$$

A simulated sample of the load from a given probability field ($\alpha = 0.03781$) is shown in Fig. 5.2. We will assume that the linear elasticity provides reliable results for all loads under consideration. The solution of our model problem is a stochastic function with the mean being the (deterministic) solution for the mean load.

Our aim is to determine the variance and covariance of the values of interest.

Concerning the failure criterion we will assume as example:

a) The criterion of stress intensity factor F in the point C (see Fig. 5.1).

b) The failure criterion based on the envelope of the Mohr's circles in the point D (see Fig. 5.1).

Knowing the stress intensity factor as random variable characterized by its mean and standard deviation, we can establish the probability level of the failure when the material probabilistic characterization of admissible stress intensity factor is given.

The criterion b) is more complicated. We need here the correlation of the components of the stress tensor which allows us to compute not only the mean Mohr's circles but also its perturbation in every point of the circle which for a given probability level has an elliptic character. The envelope of these ellipses will be compared with the admissible failure curve (see Fig. 5.3).

The concrete computation of our model problem has been computed using the program PROBE and the mesh shown in Fig. 5.4. (The refinement in the neighborhood of the reentrant corners is not shown.)

In Table 5.1 we show the mean values and the standard deviation Sd(F) of the stress intensity factor F in the point C in dependence on p.

For the technique used in PROBE for the computation of the stress intensity factor, see [8].

TABLE 5.1

p	F	sd(F)
1	-46.5958	3.71714
2	-51.7433	3.92931
3	-49.3796	3.94039
4	-49.0721	3.91575

The Mohr' circles for the probability level 90% are shown in Fig. 5.5.

5.2. The problem of stochastic boundary

The problem of stochastic boundary is more complicated but it can be transformed to the case of stochastic load. Let us consider—for simplicity of the exposition--the problem of a symmetric, cracked panel (plane strain, $\nu = 0.3$) shown in Fig. 5.6 and assume that the deterministic traction T at the boundary is such that the exact stress tensor is given by the following formulae

$$(5.2)\qquad \begin{aligned} \sigma_x &= (2\pi r)^{-1/2} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \sigma_y &= (2\pi r)^{-1/2} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \tau_{xy} &= (2\pi r)^{-1/2} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} . \end{aligned}$$

These functions are symmetric mode functions of the stress intensity factor. Let us assume that the side A is perturbed by a stochastic perturbation so that the boundary is given by the function $y = 1 + \Lambda(x,\omega)$, $-1 < x < 1$, $\Lambda(\pm 1,\omega) = 0$, where $\Lambda(x,\omega)$ is the stochastic function with the correlation function $K_1(x_1,x_2)$.

We use in our model problem

$$(5.3)\qquad K_1(x_1,x_2) = f\left(\frac{|x_1-x_2|}{2}\right) - f\left(\frac{x_1+x_2}{2} + 1\right), \qquad |x_i| < 1, \quad i = 1,2,$$

where

$$f(\xi) = \frac{8-15\xi^2(2-\xi)^2}{720}.$$

A simulated sample of the perturbance is shown in Fig. 5.7. Our aim is to find the stress intensity factor F and its standard deviation $Sd(F)$ caused by the random boundary and the stresses and their variances and covariances in $(0.1, 0.9)$.

Before addressing this problem, we have to know how the traction will change when the domain is changing so that the equilibrium is always guaranteed. We will assume to this end that functions $\tilde{\sigma}_x(x,y)$, $\tilde{\tau}_{xy}(x,y)$ and $\tilde{\sigma}_y(x,y)$ are defined in the neighborhood of the side A-B such that

$$\text{(5.4)} \qquad \begin{aligned} \frac{\partial\tilde{\sigma}_x}{\partial x} + \frac{\partial\tilde{\tau}_{xy}}{\partial y} &= 0 \\ \frac{\partial\tilde{\tau}_{xy}}{\partial x} + \frac{\partial\tilde{\sigma}_y}{\partial y} &= 0 \end{aligned}$$

and $\tilde{\sigma}_y = \sigma_y$ and $\tilde{\tau}_{xy} = \tau_{xy}$ on A-B where (τ_{xy}, σ_y) is the given traction vector at A-B. If now point D lies on the perturbed boundary, then the traction vector T is

$$\text{(5.5)} \qquad T = \begin{bmatrix} \tilde{\sigma}_x, & \tilde{\tau}_{xy} \\ \tilde{\tau}_{xy}, & \tilde{\sigma}_y \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where (n_1, n_2) is the outer normal to the perturbed boundary. This guarantees the equilibrium condition for every perturbation.

Assume that the magnitude of the perturbance is λ. Then we have

THEOREM 5.1. The solution of (up to higher order terms in λ) the perturbed problem is the solution of the original domain with the modified load T

$$(5.6) \qquad T = T_0 - \left[\Lambda(x,\omega) \frac{\partial}{\partial y} \begin{bmatrix} -\tilde{\tau}_{xy} + \tau_{xy} \\ \\ -\tilde{\sigma}_y + \sigma_y \end{bmatrix} + \Lambda'(x,\omega) \begin{bmatrix} \tilde{\sigma}_x + \sigma_x \\ \\ -\tilde{\tau}_{xy} + \tau_{xy} \end{bmatrix} \right]$$

where $T_0 = (\tau_{xy}, \sigma_y)^T$ is the traction on A-B. ■

Theorem 5.1 gives us immediately the possibility to solve the problem in the same vein as in the previous section.

We have used in our model problem

$$\tilde{\sigma}_x(x,y) = \sigma_x(0,1)$$

$$\tilde{\tau}_{xy}(x,y) = \tau_{xy}(x,1)$$

$$\tilde{\sigma}_y(x,y) = \sigma_y(x,1) - (y-1) \frac{\partial \tau_{xy}}{\partial x}(x,1).$$

We used the program PROBE and for $p = 8$ we obtained

a) The stress intensity factor F:

the mean value $F = 0.99830$ (exact value $F = 1$)

the standard deviation $sd(F) = 2.54(-4)$.

b) The stress in the point $A = (0.1, 0.9)$:

the mean value: $\bar{\sigma}_x = 0.1426$, $\bar{\sigma}_y = 0.4821$, $\bar{\tau}_{xy} = 0.1206$

the standard deviation $sd(\sigma_x) = 0.48418(-2)$, $sd(\sigma_y) = 0.35178(-2)$, $sd(\tau_{xy}) = 0.2088(-2)$

the covariance $c(\sigma_x, \sigma_y) = 0.1697(-4)$, $c(\sigma_x, \tau_{xy}) = 0.9493(-5)$ $c(\sigma_y, \tau_{xy}) = 0.7019(-5)$

the normalized covariance $\rho(\sigma_x, \sigma_y) = 0.9966$, $\rho(\sigma_x, \tau_{xy}) = 0.9389$, $\rho(\sigma_y, \tau_{xy}) = 0.9555$.

We see that the variance of F is much smaller than the variance of the stress in the point A. If the failure criterium is based on the stress intensity factor F, then it is (in our case) practicaly uninfluenced by the uncertainty of the boundary. If the failure creterium is based on the Mohr circle in A, then it is much more sensitive to the uncertainty of the boundary. This shows very clearly that the same uncertainties can lead to the uncertainties of different magnitude in the failure criterium parameters.

Let us mention that we need in (5.6) the derivatives of the stresses of the (deterministic) solution which is computed by the finite element method. This, of course, needs a special care and the computation can be made by the postprocessing technique (see [8]).

The selection of functions $\tilde{\sigma}_x$, σ_y, $\tilde{\tau}_{xy}$ does not make usually any problems. Many times we have a traction free surface and then, of course, $\tilde{\sigma}_x = \tilde{\sigma}_y = \tilde{\tau}_{xy} = 0$ is the proper choice.

We have assumed that the tractions are not stochastic. We can also treat the combined case when both the domain and the traction are stochastic.

We have shown here only illustrative examples of relatively simple structure. The theory and implementation principles were developed for the general case. It is possible to compute also higher correlation functions and obtained, e.g. the skewness of the distribution of the stress intensity factors, etc.

In the case of stochastic material coefficients, we can proceed similarly and get by an iterative technique the desired data for small variation of the material coefficients.

The optimal design should in general take into account the stochastic character of the input data.

6. CONCLUSIONS

Solving the problems of the optimal design and the engineering problems in general one has to take into account various aspects of the mathematical model and its numerical treatment for getting reliable results. Detailed a-priori mathematical analysis is of utmost importance for the reliable conclusions.

REFERENCES

[1] Babuška, I., The continuity of the solutions of elasticity problems on small deformation of the region, ZAMM 39 (1959), 411-412. (German)

[2] Babuška, I., On randomized solution of Laplace's equation, Casopis Pest. Mat. 86 (1961), 269-276.

[3] Babuška, I., The stability of the domain of definition with respect to basic problems of the theory of partial differential equations especially with respect to the theory of elasticity I, II, Czechoslovak Math. J. 11 (1961), 76-105, 165-203. (Russian)

[4] Babuška, I., The theory of small change in the domain of definition in the theory of partial differential equations and its applications, Proc. of the Conf. EQUADIFF, Prague (1962), 13-26.

[5] Babuška, I., The stability of domains and the question of the formulation of plate problems, Apl. Mat. 7 (1962), 463-467. (German)

[6] Babuška, I., Szabo, B. A., Katz, I. N., The p-version of the finite element method, SIAM J. Numer. Anal. 18 (1981), 515-545.

[7] Babuška, I., Szabo, B. A., On the rates of convergence of th finite element method, Internat. J. Numer. Methods Engrg. 18 (1982), 323-341.

[8] Babuška, I., Miller, A., The Post-processing Approach in the Finite Element Method, Part I, II, III, Internat. J. Numer. Methods Engrg. 20 (1984), 1085-1109, 1111-1129, 2311-2325.

[9] Babuška, I., Suri, M., The optimal convergence rate of the p-version of the finite element method. To appear.

[10] Banichuk, N. V., Problems and methods of optimal structural design, Plenum Press, New York, London, 1983.

[11] Ciarlet, P. G. Destuynder, P., A justification of the two dimensional linear plate model. J. Mécanique 18 (1979), 315-344.

[12] Ciarlet, P. G., Rabier, P., Les Equations de von Karmán, Lecture Notes in Mathematics No. 826, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[13] Gallagher, R. H., Atrek, E., Ragsdell K., Zienkiewicz, D. C., eds. Optimum Structural Design, J. Wiley and Sons, 1983.

[14] Guo, B., Babuška, I., The h-p version of the finite element method, Tech. Note BN-1043, Institute for Physical Science and Technology, University of Maryland, July 1985

[15] Kantorovich, L. V., Krylov, V. I., Approximate Methods of Higher Analysis, Noordhoff Groningen, 1958.

[16] Kohn, R. V., Vogelius, M., A new model for thin plates with rapidly varying thickness, Internat. J. Solids and Structures 20 (1984), 333-350.

[17] Kohn, R. V., Vogelius, M., Thin plates with rapidly varying thickness and their relation to structural optimization, IMA Preprint Ser. 155, Institute for Mathematics and its Applications, University of Minnesota, June 1985.

[18] Kohn, R. V., Strang, G., Optimal design and relaxation of variational problems, to appear in Comm. Pure. Appl. Math.

[19] Larsen, S., Numerical Analysis of Elliptic Partial Differential Equations with Stochastic Input Data, Ph.D. Dissertation, University of Maryland, 1985.

[20] Morgenstern, D., Herleitung der Plattentheorie aus der dreidimensionalen Elasticitätstheorie, Arch. Rational Mech. Anal. 4 (1959), 145-152.

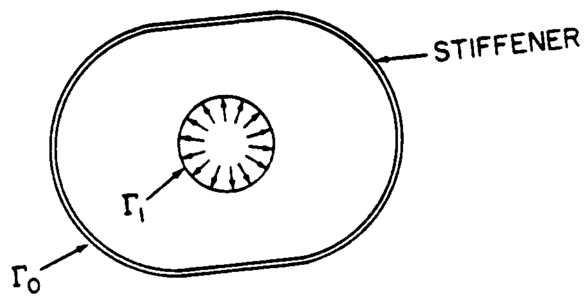
[21] Pironneau, D., Optimal shape design for elliptic systems, Springer-Verlag, New York, 1984.

[22] Szabo, B. A., Mesh Design for the p-version of the Finite Element Method, Rep. No. WU/CCM-85/2, Center for Computational Mechanics, Washington University, St. Louis, 1985.

[23] Szabo, B. S., Implementation of a Finite Element Software System with h- and p-Extension Capabilities, Proc. 8th Invitational UFEM Symposium: Finite Element Software Systems, ed. H. Kardestuncer, University of Connecticut, 1985.

[24] Szabo, B. A., PROBE: Theoretical Mannual, Noetic Technologies Corporation, St. Louis, Missouri, 1985.

FIGURES



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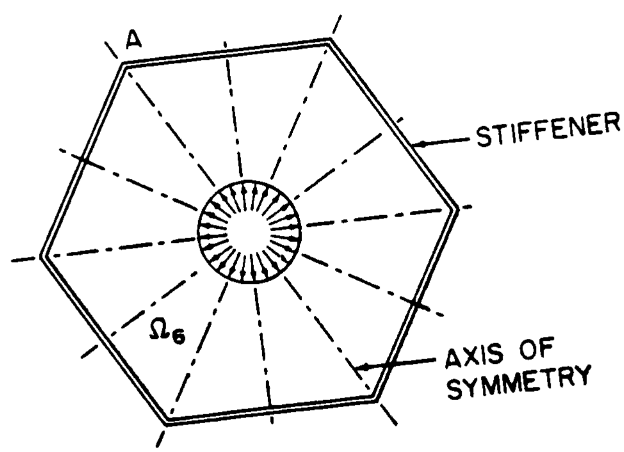
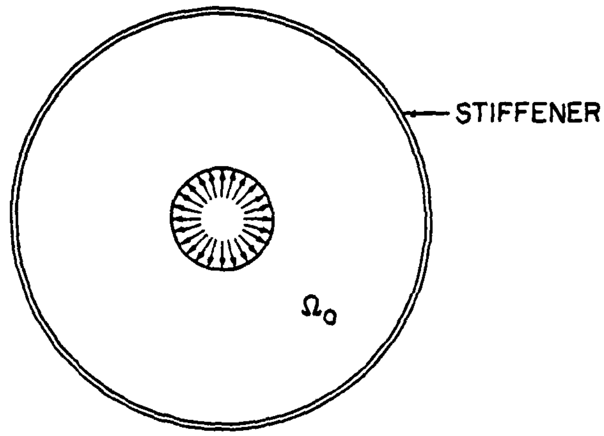


Figure 2.2



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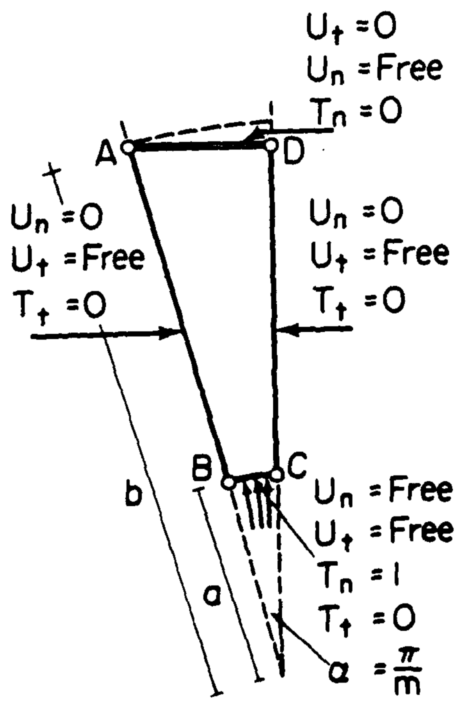

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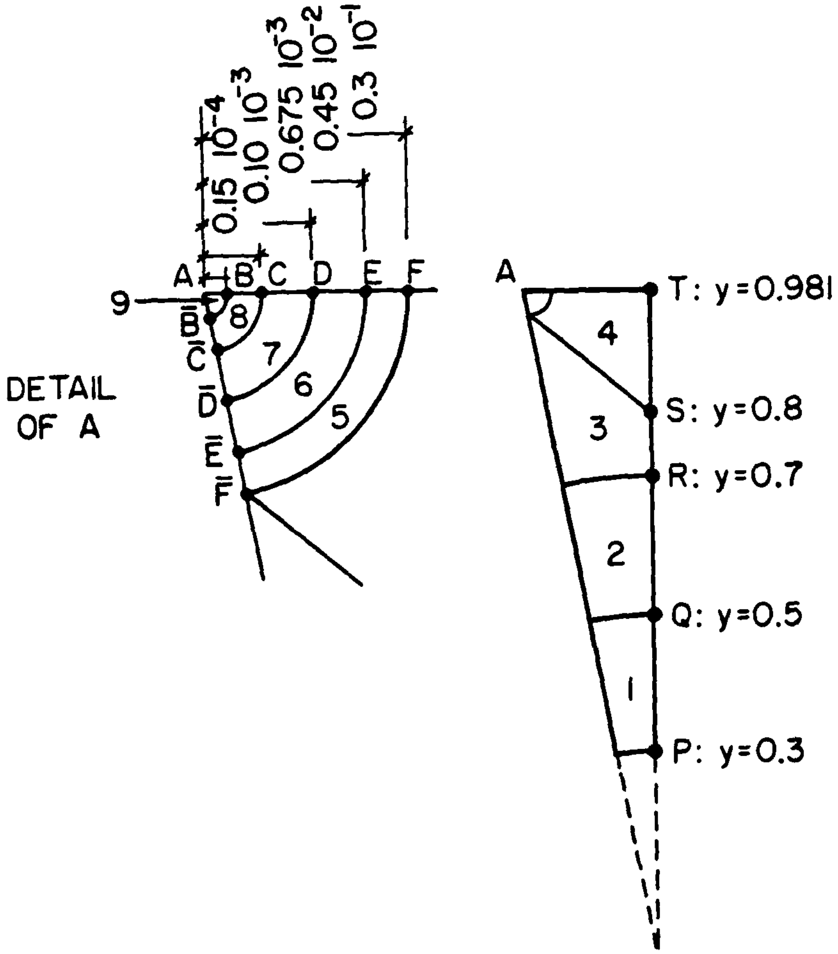


0.15 10^-4
0.10 10^-3
0.675 10^-3
0.45 10^-2
0.3 10^-1
A B C D E F
9
8
7
6
5
DETAIL
OF A
A
T: y=0.981
S: y=0.8
R: y=0.7
Q: y=0.5
P: y=0.3
4
3
2
1

Figure 2.5

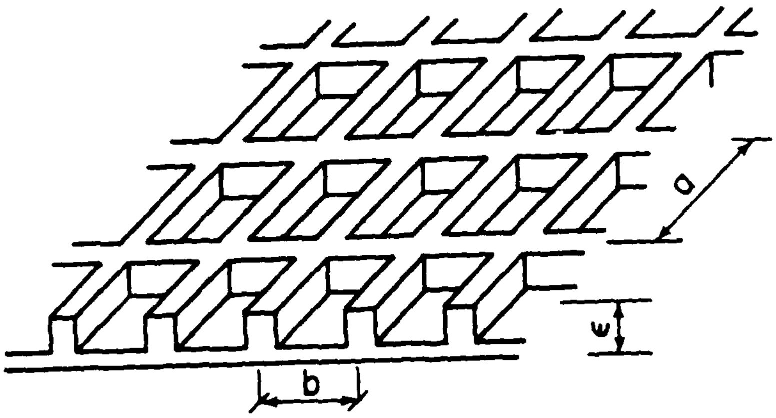


Figure 3.1

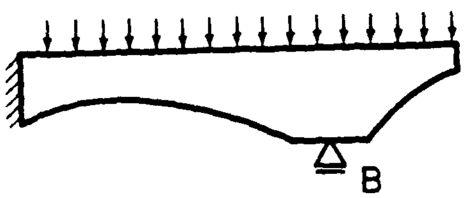
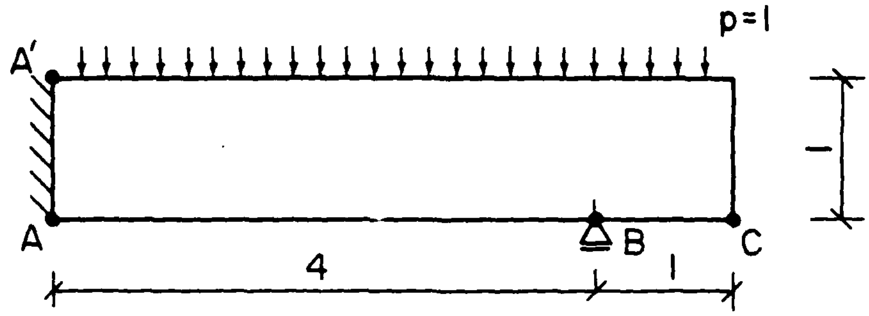


Figure 4.1



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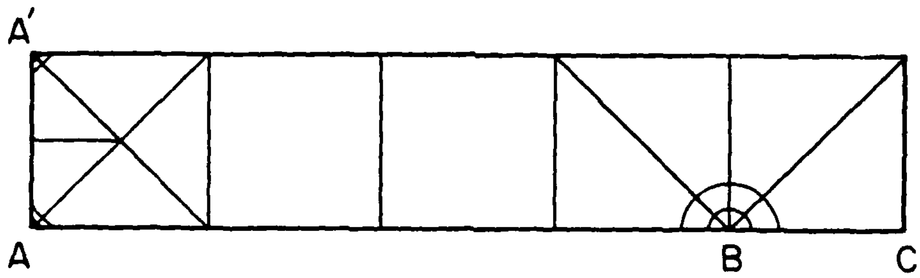


Figure 4.3

DETAIL B

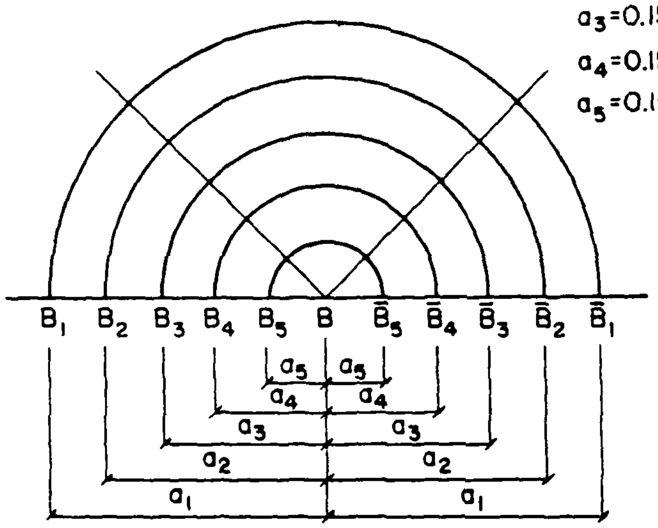
$a_1 = 0.15$

$a_2 = 0.15^2 = 0.225 \ 10^{-1}$

$a_3 = 0.15^3 = 0.3375 \ 10^{-2}$

$a_4 = 0.15^4 = 0.5063 \ 10^{-3}$

$a_5 = 0.15^5 = 0.7594 \ 10^{-4}$



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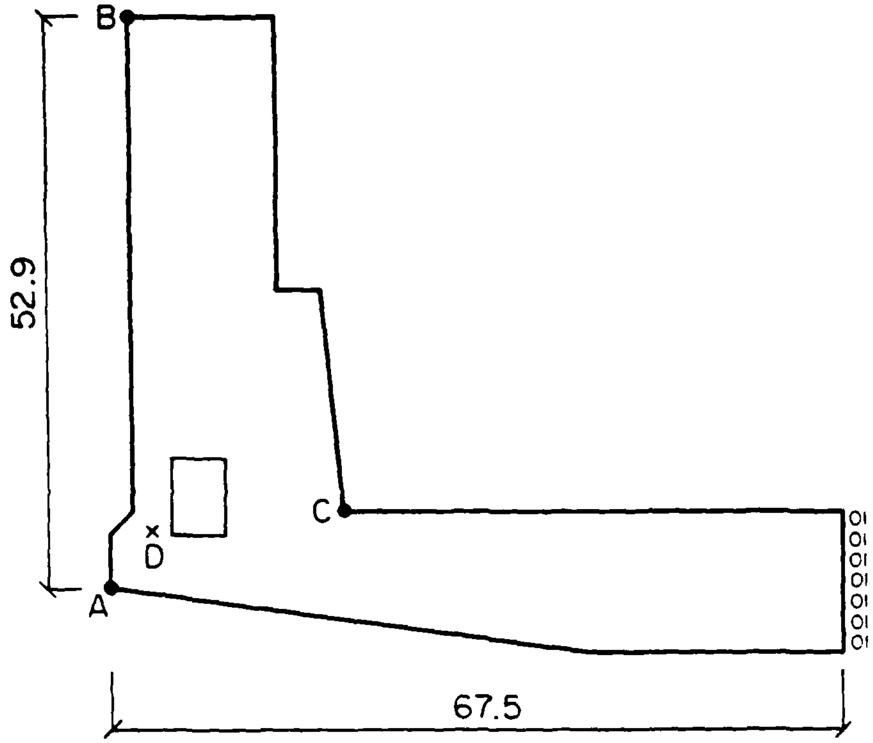

Figure 5.1

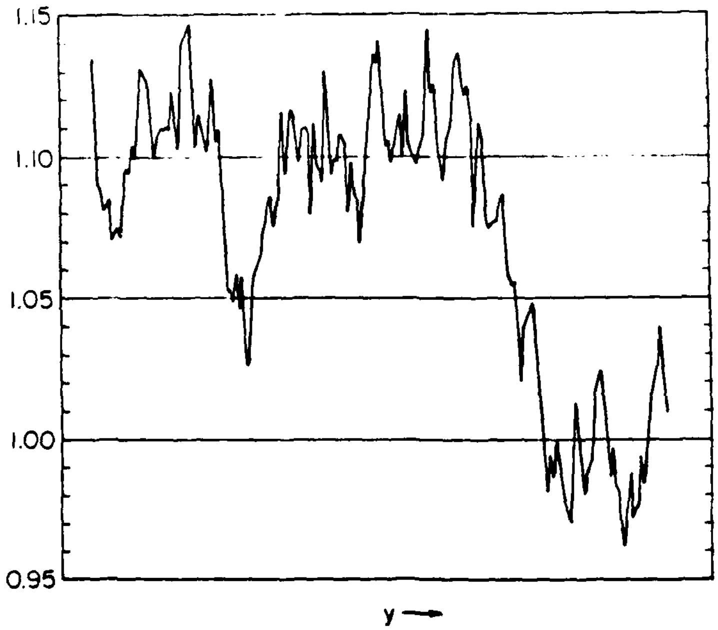

Figure 5.2

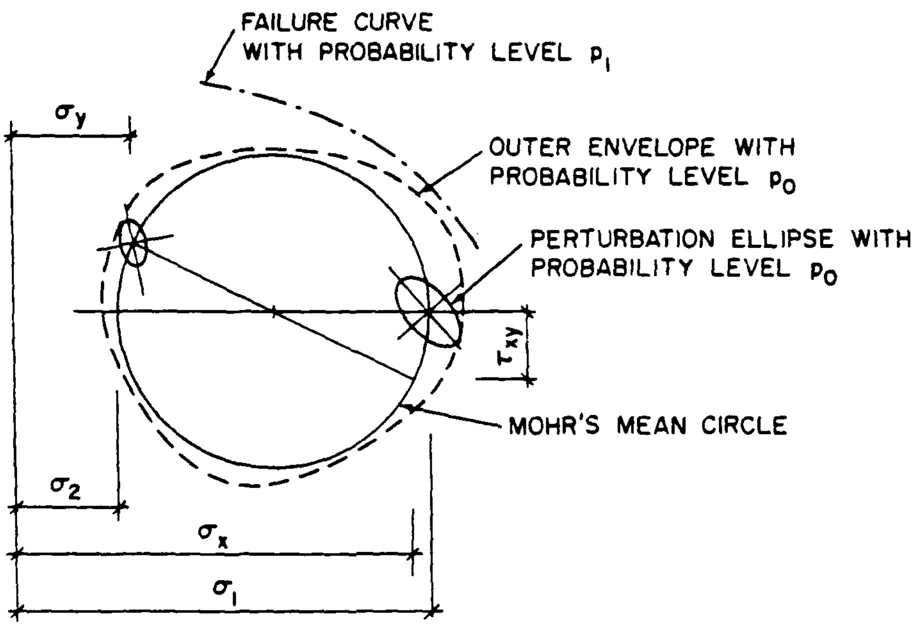
FAILURE CURVE
WITH PROBABILITY LEVEL p_1
σ_y
OUTER ENVELOPE WITH
PROBABILITY LEVEL p_0
PERTURBATION ELLIPSE WITH
PROBABILITY LEVEL p_0
τ_{xy}
MOHR'S MEAN CIRCLE
σ_2
σ_x
σ_1

Figure 5.3

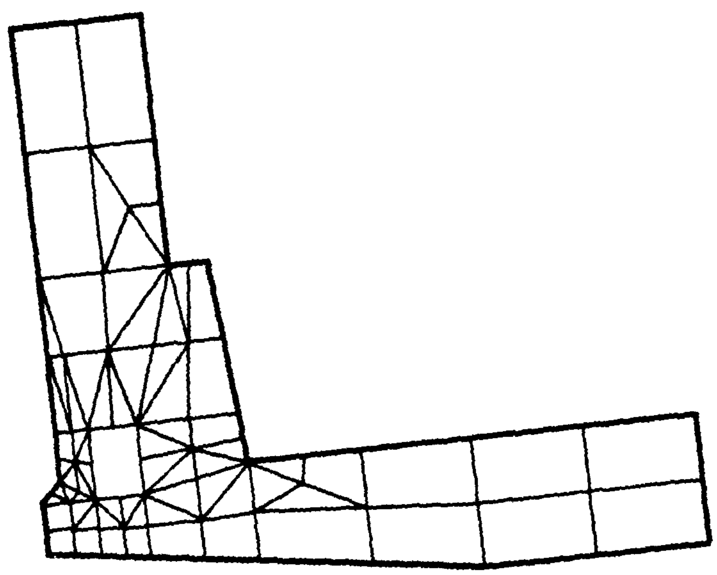

Figure 5.4

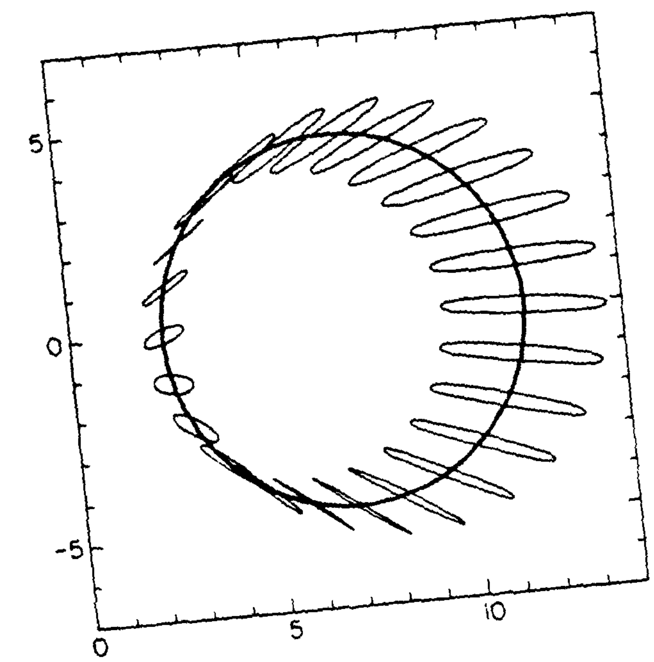

Figure 5.5

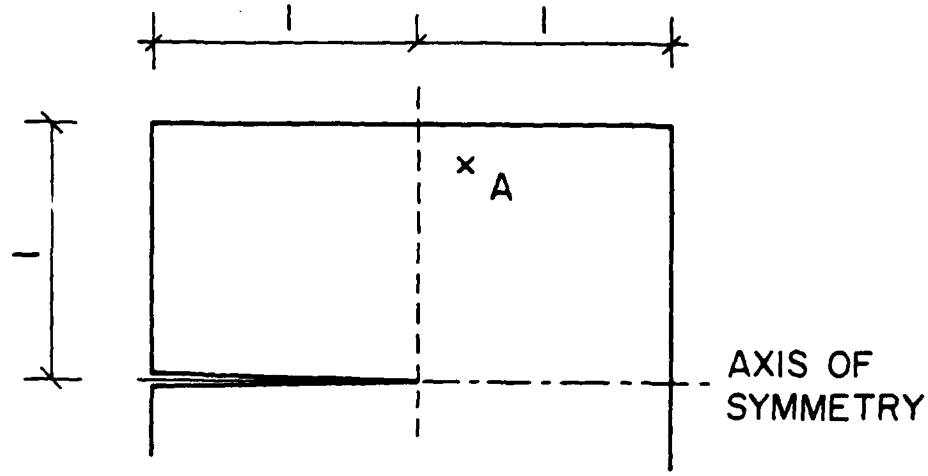

Figure 5.6

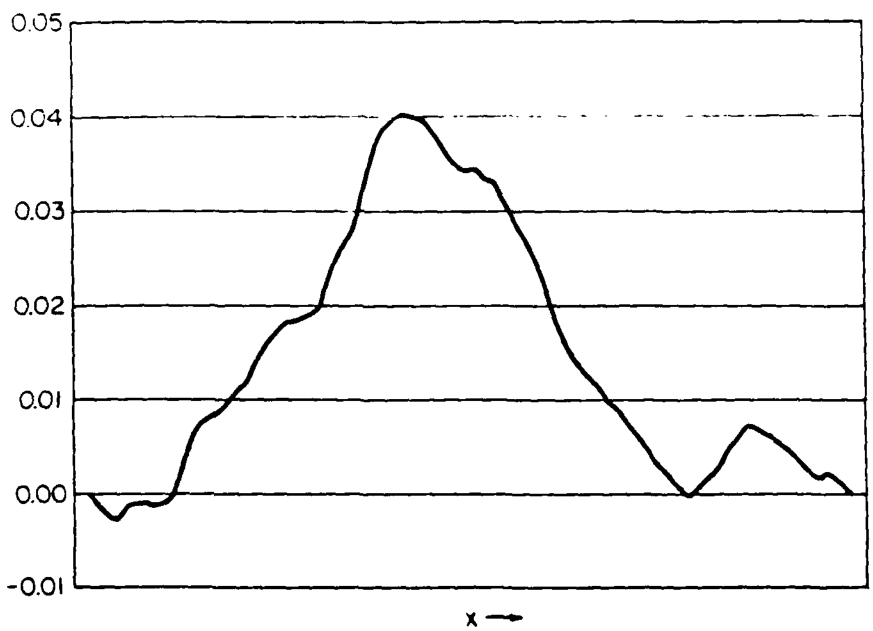

Figure 5.7

<u>The Laboratory for Numerical analysis</u> is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

<u>Further information</u> may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

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